

MIDTERM EXAMINATION

FORM

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1. The following function, f , maps the real line into itself. $f: \mathbb{R} \rightarrow \mathbb{R}$. We must decide if it is continuous at $x = 0$. Please choose between two proposed answers below. Explain your choice.

$$\begin{aligned} \text{For } x < 0, f(x) &= x \\ x = 0, f(x) &= 0 \\ x > 0, f(x) &= x + 2. \end{aligned}$$

Proposed answer A: Yes the function is continuous at 0. For any $\varepsilon > 0$, choose $\delta = .5\varepsilon$. Then for any $x < 0$, $|0 - x| < \delta$, $|f(x) - f(0)| = |x - 0| < \delta = .5\varepsilon < \varepsilon$. This is the definition of continuity.

Proposed answer B: No the function is not continuous at 0. It makes a jump in the neighborhood of $x > 0$. For $2 > \varepsilon > 0$, there is no $\delta > 0$, such that for $x > 0$ with $|x - 0| < \delta$, it follows that $|f(x) - f(0)| < \varepsilon$. Instead we have $|f(x) - f(0)| \geq 2 > \varepsilon$.

Problems 2, 3, and 4 below treat allocation in an Edgeworth Box. There are two households, denoted 1 and 2, two goods denoted x and y . The two households have identical preferences denoted by the utility function $U(x, y) = x \cdot y$. For any consumption plan (x, y) , with x and $y > 0$, the household's marginal rate of substitution,

$$\text{MRS}_{x,y} = \frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} = \frac{U_x}{U_y} = \frac{y}{x}.$$

Household 1's endowment is $r^1 = (10, 2)$; household 2's endowment is $r^2 = (2, 10)$.

2. The Walrasian auctioneer proposes the price vector

$p = (p_x, p_y) = (0.5, 0.5)$. In order for this to be a competitive equilibrium price vector there must be a consumption plan (x^1, y^1) , (x^2, y^2) for each household that fulfills budget constraint, optimizes utility subject to budget constraint, and clears the market. Consider $(x^1, y^1) = (x^2, y^2) = (6, 6)$. Demonstrate that this allocation is a competitive equilibrium.

3. The First Fundamental Theorem of Welfare Economics says that a competitive equilibrium allocation is Pareto efficient. Demonstrate that the allocation $(x^1, y^1) = (x^2, y^2) = (6, 6)$ is Pareto efficient.

4. Draw the Edgeworth Box, the contract curve, and the equilibrium budget line. Describe and explain.

5. The Brouwer Fixed Point Theorem can be applied to the closed unit interval in \mathbb{R} , $[0, 1] = \{x \mid x \in \mathbb{R}, 0 \leq x \leq 1\}$. Then we say, *Let $f: [0, 1] \rightarrow [0, 1]$, f continuous on $[0, 1]$, then there is x^* in $[0, 1]$ so that $f(x^*) = x^*$.*

Demonstrate that this statement is not true (that is, the fixed point may not exist) when we revise the domain and range to the open unit interval $(0, 1) = \{x \mid x \in \mathbb{R}, 0 < x < 1\}$. That is, show that the following statement is false: *Let $f: (0, 1) \rightarrow (0, 1)$, f continuous on $(0, 1)$, then there is x^* in $(0, 1)$ so that $f(x^*) = x^*$.*

A counterexample with an explanation is sufficient.

Hint: Consider $f(x) = .5 + .5x$. $f: (0, 1) \rightarrow (0, 1)$. Use a proof by contradiction. Suppose there is $x^* = f(x^*) = .5 + .5x^*$. Then $.5x^* = .5$, and $x^* = 1$. Is this a contradiction? Explain.